The boundary layer on a magnetized plate

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(Received 24 October 1961)

This paper is concerned with the magnetohydrodynamic boundary layer in uniform flow past a flat plate, when a uniform magnetic field in the stream direction is applied at the plate. For both large and small values of the electrical conductivity parameter, solutions in series are derived for the velocity and magnetic fields. In each of the cases it is shown that if the strength of the applied field exceeds a certain critical value, boundary-layer separation occurs.

1. Introduction

The aim of the present investigation is to gain further insight into the fundamentals of the flow of a conducting fluid of small viscosity past a solid body. Solutions for the motion of a perfectly conducting inviscid fluid are obtainable, but it is difficult to know how much trust to put in these in view of the frequently disastrous consequences of omitting viscosity in the analysis of a non-conducting fluid. For the latter, experimental observation and boundary-layer theory have over the years built up a very satisfactory background by which to judge whether a particular inviscid solution can be relied upon, or whether separation will cause a gross change in the flow. For conducting fluids observation has as yet virtually nothing to offer. The remaining hope seems to be to build up understanding through studying magnetohydrodynamic boundary layers, starting with the flows which are most tractable mathematically.

The problem discussed in this paper is the steady two-dimensional flow of an incompressible viscous electrically conducting fluid past a semi-infinite flat plate, when both the Reynolds number and the magnetic Reynolds number are large. There is no magnetic field in the distant fluid, but in the boundary layer there is a field in the stream direction, generated by external means within the plate itself. The streamwise component of the field at the plate surface is constant. The electric field is zero everywhere. One physical interest in this flow lies in the possibility of using such a field to shield a body from excessive heating. Across a magnetohydrodynamic boundary layer the sum of the static and magnetic pressures is constant, and this means that the pressure at the surface is reduced by the amount of the magnetic pressure of the applied field. For a compressible fluid this would imply an equivalent change in density, and hence in heat transfer.

The mathematical technique used is a development of that given in a previous paper by the author (Glauert 1961, hereafter referred to as G), which treated a similar flow but with the basic magnetic field in the fluid instead of in the plate. The essential idea is that when the conductivity parameter ϵ, the ratio of the viscous and magnetic diffusivities, is either large or small, the boundary layer may be treated as two largely separate layers, and the required solution may be built up in series by suitably matching conditions far out in the inner layer with those close in in the outer layer. The method requires care in its application, as will become apparent, since more than one form of transformation of variables may at first sight appear feasible. The justification of a successful application of the method is that it can be seen how the next terms in the series expansions may be determined, and determined uniquely. It is not easy to show that the series can be continued indefinitely in any specific manner. Thus logarithmic terms are frequently required at certain stages, and later on perhaps 'log log' terms would be also needed.

The two cases of ϵ large and small compared with unity are referred to as those of large and small conductivity respectively. As in G, the requirement that the magnetic Reynolds number be large limits the applicability of the results for small conductivity. In this paper a further subdivision is necessary. When the field strength parameter β, to be defined in equation (2.7), is sufficiently small the leading terms of the series solutions are independent of β, but when β is larger even the leading terms depend intimately on β, and the forms of the expansions are modified. We describe the fields in these two cases as being weak and strong respectively.

The most striking result of the analysis is that for both large and small conductivity the boundary layer separates from the surface for sufficiently large values of the applied field strength. This confirms the importance of magnetically induced separation as an effect to be reckoned with in magnetohydrodynamics.

The possibility of setting up physically the assumed field requires examination. Consider the flow past a non-magnetic body of revolution in which, after a short rounded or conical nose, there is a substantial length of uniform circular section. Over this length the stream velocity is uniform, and as far as the boundary layer is concerned it may justifiably be treated as a flat-plate flow. Suppose now that there is a closely wound solenoid round this cylindrical section, embedded so as to leave the surface smooth. In magnetostatics a current passing through this solenoid produces a strong magnetic field inside the cylinder but only a weak one outside, since field lines emerging from the two ends of the coil join up by circuits extending to large distances. In magnetohydrodynamics the situation is fundamentally different. The field lines which emerge at the upstream end cannot escape, but are swept back by the stream and concentrated into a boundary layer outside the cylinder, producing a correspondingly strong field there.

Suppose that the field strengths just outside and inside the coil are $H_0\mathbf{i}$ and $H_1\mathbf{i}$ respectively, where $\mathbf{i}$ is a unit vector parallel to the axis of the body. If the boundary layer is thin there can be no comparable field in the radial direction. The equation $\mathbf{j} = \operatorname{curl}\mathbf{H}$ (in MKS units) shows that

$$H_0 - H_1 = J, \tag{1.1}$$

where J is the total current in the solenoid per unit distance in the axial direction. If the cylinder is long, the axial field inside is effectively constant across the section. In the boundary layer the flux of $\mathbf{H}$, per unit distance in the circumferential direction, is $H_0\delta$, where δ is a measure of the magnetic-boundary-layer thickness. Since $\operatorname{div}\mathbf{H} = 0$, the total flux across the section must be zero, and hence, if d is the cylinder diameter,

$$\pi d\delta H_0 + \tfrac{1}{4}\pi d^2 H_1 = 0,$$

or
$$H_1 = -4(\delta/d)\,H_0. \tag{1.2}$$

This shows that when δ/d is small, H_1 is small compared with H_0. In this case, from (1.1),
$$H_0 \simeq J. \tag{1.3}$$

This calculation proves that a uniformly wound solenoid would give the necessary magnetic field. There is also a small component of field across the solenoid. This is needed to provide the additional flux in the boundary layer as δ increases, and causes a corresponding change in H_1, as given by (1.2).

The basic equations for the problem of this paper were written down by Zhigulev (1959*a*), but he obtained no solutions. Zhigulev also considered the effects of a magnetic field in the circumferential direction, as would be produced by electric current flowing back in the boundary layer from an electrode at the nose of the body. This has no effect at all on the velocity distribution; it produces only a pressure gradient normal to the surface, and the distribution of magnetic intensity within the boundary layer is given by simple integration. Rossow (1957) considered the flow past a flat plate in a transverse magnetic field. He distinguished two cases, the field fixed in the fluid and the field fixed in the plate. This latter case might by thought to be related to our problem, but in fact the analysis is in error since the electric field is ignored. Rossow's field extends indefinitely, instead of being confined to a boundary layer, and this cannot be correct for a field generated within the plate.

Meyer (1960) has studied the problem analogous to that treated here for the axi-symmetric flow near a stagnation point. He concentrates on the case of small ϵ, as being the one of chief physical interest, and proceeds to set up inner and outer solutions, and to match them as in the present paper in the limiting case $\epsilon \to 0$. He evaluates the solutions numerically for a variety of values of the applied field strength, but does not detect any tendency to separation.

Since a negative pressure at the plate is inadmissible physically, the change in magnetic pressure $\frac{1}{2}\mu H_0^2$ across the boundary layer cannot exceed the stream pressure p_0. Any stronger field must cause the whole flow to leave the surface. This phenomenon, a form of cavitation, has been discussed by Zhigulev (1959*b*) and is quite distinct from the separation encountered in this paper. Here, as in ordinary incompressible hydrodynamics, p_0 does not enter into the equations. For an actual fluid, the question of whether one would expect cavitation or separation to occur first as the field strength is increased depends on the particular values of p_0 and ϵ. Plausible physical situations can be envisaged for each possibility.

2. Large conductivity, weak field

The equations which govern the flow in the boundary layer (derived precisely as in **G**) are

$$f''' + ff'' - \beta gg'' = 0, \tag{2.1}$$

$$g'' + \epsilon(fg' - f'g) = 0, \tag{2.2}$$

with boundary conditions

$$f(0) = f'(0) = 0, \quad f'(\infty) = 2, \quad g'(0) = 2, \quad g(\infty) = 0. \tag{2.3}$$

The only change from the problem treated in **G**, with the basic field in the fluid rather than in the plate, lies in the boundary conditions on g. The independent variable η is given by

$$\eta = \tfrac{1}{2}(U_0/\nu x)^{\frac{1}{2}}\, y, \tag{2.4}$$

x and y being distances along and perpendicular to the plate, measured from the leading edge. The velocity and magnetic field components are

$$u = \tfrac{1}{2}U_0 f', \quad v = \tfrac{1}{2}(U_0\nu/x)^{\frac{1}{2}}\,(\eta f' - f), \tag{2.5}$$

$$H_x = \tfrac{1}{2}H_0 g', \quad H_y = \tfrac{1}{2}H_0(\nu/U_0 x)^{\frac{1}{2}}\,(\eta g' - g). \tag{2.6}$$

The fluid has density ρ, kinematic viscosity ν, electrical conductivity σ and magnetic permeability μ. The speed of the undisturbed stream is U_0 and the magnetic field intensity at the plate has x-component H_0. Also

$$\epsilon = \sigma\mu\nu, \quad \beta = \mu H_0^2/\rho U_0^2. \tag{2.7}$$

It is no longer possible, as in **G**, to interpret β as the square of the Alfvén speed, since outside the boundary layer there is no field. Nevertheless, it is convenient to retain the parameter β in its previous form, as a suitable non-dimensional measure of the strength of the applied field. The conductivity parameter ϵ is equal to the ratio of the viscous and magnetic diffusivities.

When ϵ is large we expect the boundary layer to consist of two parts, an outer one where viscous forces are important but magnetic diffusion is negligible, and an inner one governed largely by magnetic forces.

We may hope to obtain the first approximations $f_0(\eta)$, $g_0(\eta)$ applicable in the outer layer by ignoring the first term in (2.2) and discarding the boundary condition on $g'(0)$. The required solutions are

$$f_0 = B(\eta), \quad g_0 = 0, \tag{2.8}$$

where $B(\eta)$ is the Blasius function governing the boundary layer on a flat plate in a non-conducting fluid. Final approval of these solutions must await successful matching with the inner solutions.

In the inner layer it is necessary to employ modified equations. The terms of (2.2) must be comparable, g' must be $O(1)$, and since $B''(0) = A = 1{\cdot}3282$, f'' must be $O(1)$. Accordingly we write

$$\eta = \epsilon^{-\frac{1}{3}}\xi, \quad f(\eta) = \epsilon^{-\frac{2}{3}}F(\xi), \quad g(\eta) = \epsilon^{-\frac{1}{3}}G(\xi). \tag{2.9}$$

Thus $f''(\eta) = F''(\xi)$, $g'(\eta) = G'(\xi)$, etc., and (2.1) and (2.2) become

$$F''' + \epsilon^{-1}FF'' - \beta\epsilon^{-\frac{1}{3}}GG'' = 0, \tag{2.10}$$

$$G'' + FG' - F'G = 0, \tag{2.11}$$

with boundary conditions

$$F(0) = F'(0) = 0, \quad G'(0) = 2. \tag{2.12}$$

When ξ is large there must be suitable agreement with the outer solutions with η small. These regions correspond since ϵ is large. We may note that the relation between g and G differs from that required at the corresponding stage of the analysis in **G**.

We have already assumed that ϵ^{-1} is small, but to proceed with the solution of (2.10) a further assumption is necessary. Suppose first that the field is sufficiently weak for $\beta\epsilon^{-\frac{1}{3}}$ to be small compared with unity. What happens when the field is stronger will be considered in § 3.

For the first approximations, we see from (2.10), (2.12) and the matching requirement with $f''(0)$ that

$$F_0 = \tfrac{1}{2}A\xi^2, \tag{2.13}$$

and hence (2.11) becomes

$$G_0'' + \tfrac{1}{2}A\xi^2 G_0' - A\xi G_0 = 0. \tag{2.14}$$

The required solution is

$$G_0 = 2\xi\,{}_1F_1(-\tfrac{1}{3}, \tfrac{4}{3}, -\tfrac{1}{6}A\xi^3) + K\,{}_1F_1(-\tfrac{2}{3}, \tfrac{2}{3}, -\tfrac{1}{6}A\xi^3), \tag{2.15}$$

where ${}_1F_1$ is the confluent hypergeometric function and the constant

$$K = -\tfrac{4}{3}(6/A)^{\frac{1}{3}}[(\tfrac{1}{3})!/(\tfrac{2}{3})!]^2 = -2{\cdot}156. \tag{2.16}$$

This is chosen so that G_0 is exponentially small for large ξ, giving the necessary matching with g_0. We see that $G_0(0) = K$.

These solutions may now be improved by including the effects of small but non-zero values of $\beta\epsilon^{-\frac{1}{3}}$ and ϵ^{-1}. As in **G**, the extra contributions are found by examining the unsatisfied terms in the equations and the imperfections in matching between the inner and outer layers. In the outer layer f_0, g_0 are exact solutions of equations (2.1) and (2.2). For η small,

$$f_0 = \tfrac{1}{2}A\eta^2 - \tfrac{1}{120}A^2\eta^5 + \ldots,$$

which implies that, for ξ large,

$$F \simeq f_0\epsilon^{\frac{2}{3}} \sim \tfrac{1}{2}A\xi^2 - \tfrac{1}{120}A^2\epsilon^{-1}\xi^5 + \ldots, \tag{2.17}$$

and so a term proportional to ϵ^{-1} must be added to the inner solution to improve the matching. Since G_0 tends to zero exponentially, neither F_0 nor G_0 calls for any modification in the outer layer. However, the form of equation (2.10) shows that perturbations proportional to both $\beta\epsilon^{-\frac{1}{3}}$ and ϵ^{-1} are needed.

Suppose that we first write

$$F = F_0 + \beta\epsilon^{-\frac{1}{3}}F_1 \tag{2.18}$$

and similarly for G, f and g. The equations and boundary conditions for the outer layer show that $f_1 = g_1 = 0$. For the inner layer, from (2.10) and (2.11),

$$F_1''' = G_0 G_0'', \tag{2.19}$$

$$G_1'' + \tfrac{1}{2}A\xi^2 G_1' - A\xi G_1 = F_1' G_0 - F_1 G_0'. \tag{2.20}$$

We require $F_1(0) = F_1'(0) = 0$, and $F_1''(\infty) = 0$ to match with f_1. This determines F_1. In particular

$$F_1''(0) = -\int_0^\infty G_0 G_0''\, d\xi. \tag{2.21}$$

With F_1 known, we can proceed to solve (2.20) with its appropriate boundary conditions $G_1'(0) = 0$, $G_1(\infty) = 0$. Numerical integration gave the values

$$F_1''(0) = -1{\cdot}220, \quad G_1(0) = -0{\cdot}657. \tag{2.22}$$

If β is very small, the chief perturbation is that in ϵ^{-1}. Let

$$F = F_0 + \epsilon^{-1} F_2, \tag{2.23}$$

and similarly for G, f and g. As before, we see that $f_2 = g_2 = 0$. The equations for F_2 and G_2 are

$$F_2''' = -F_0 F_0'' = -\tfrac{1}{2} A^2 \xi^2, \tag{2.24}$$

$$G_2'' + \tfrac{1}{2} A \xi^2 G_2' - A \xi G_2 = F_2' G_0 - F_2 G_0'. \tag{2.25}$$

The boundary conditions $F_2(0) = F_2'(0) = 0$ and the requirements that for ξ large, F_2 has a term $-\frac{1}{120} A^2 \xi^5$ (as given by (2.17)) but no term in ξ^2 (for matching with f_2) show that

$$F_2 = -\tfrac{1}{120} A^2 \xi^5. \tag{2.26}$$

Equation (2.25) can now be integrated numerically with boundary conditions $G_2'(0) = 0$, $G_2(\infty) = 0$, giving

$$F_2''(0) = 0, \quad G_2(0) = -0{\cdot}024. \tag{2.27}$$

It seems clear that, if required, further terms may be calculated in the expansions for f, g, F and G without any fundamental difficulty in matching the solutions at each stage. The results established here give for the skin friction at the plate

$$\begin{aligned} \tau_w &= \mu(\partial u/\partial y)_{y=0} = \tfrac{1}{4}\rho(U_0^3 \nu/x)^{\frac{1}{2}} F''(0) \\ &= 0{\cdot}33206\rho(U_0^3 \nu/x)^{\frac{1}{2}} \{1 - 0{\cdot}918\beta\epsilon^{-\frac{1}{3}} + \ldots\}, \end{aligned} \tag{2.28}$$

and for the normal component of magnetic intensity at the plate

$$\begin{aligned} H_n &= -\tfrac{1}{2} H_0 (\nu/U_0 x)^{\frac{1}{2}} \epsilon^{-\frac{1}{3}} G(0) \\ &= 1{\cdot}078 H_0 (\nu/U_0 x)^{\frac{1}{2}} \epsilon^{-\frac{1}{3}} \{1 + 0{\cdot}305\beta\epsilon^{-\frac{1}{3}} + 0{\cdot}011\epsilon^{-1} + \ldots\}. \end{aligned} \tag{2.29}$$

The quantity H_n is of interest since it determines the magnetic drag. By Ampère's law and (1.3), the drag force on the solenoid in the plate is $\mu H_0 H_n$ per unit area (ignoring the inaccuracy in (1.3)), and the total drag per unit area is therefore

$$\begin{aligned} T &= \tau_w + \mu H_0 H_n \\ &= 0{\cdot}33206\rho(U_0^3 \nu/x)^{\frac{1}{2}} \{1 + 2{\cdot}329\beta\epsilon^{-\frac{1}{3}} + \ldots\}. \end{aligned} \tag{2.30}$$

The skin friction decreases as β increases, and H_n/H_0 increases. If ϵ is large, the results are still applicable when β is of the order of unity, but it remains to be discussed what happens when $\beta\epsilon^{-\frac{1}{3}}$ ceases to be small.

3. Large conductivity, strong field

When $\beta\epsilon^{-\frac{1}{3}}$ is not small no change is needed in the outer layer, but in the inner layer the last term of (2.10) can no longer be ignored in the first approximation. We are compelled to revise the transformation (2.9) so that this last term is on a par with the first term. To achieve this one of our three previous requirements on the form of the transformation must be relaxed. The terms of (2.11) must continue to balance or unacceptably trivial equations will result, and the boundary condition on $g'(0)$ cannot be abandoned. Accordingly we must drop the requirement that f'' is $O(1)$ and write

$$\eta = \beta^{-\frac{1}{4}}\epsilon^{-\frac{1}{4}}\theta, \quad f(\eta) = \beta^{\frac{1}{4}}\epsilon^{-\frac{3}{4}}\phi(\theta), \quad g(\eta) = \beta^{-\frac{1}{4}}\epsilon^{-\frac{1}{4}}\psi(\theta). \tag{3.1}$$

The equations for the inner layer become

$$\phi''' + \epsilon^{-1}\phi\phi'' - \psi\psi'' = 0, \tag{3.2}$$

$$\psi'' + \phi\psi' - \phi'\psi = 0, \tag{3.3}$$

with boundary conditions

$$\phi(0) = \phi'(0) = 0, \quad \psi'(0) = 2, \quad \phi''(\infty) = A\beta^{-\frac{3}{4}}\epsilon^{\frac{1}{4}}, \quad \psi(\infty) = 0. \tag{3.4}$$

The value of $\phi''(\infty)$ is as required to match the unchanged value of $f''(0)$ from the outer layer.

For β very large, we might hope to obtain the fundamental form of solution for the inner layer by taking $\phi''(\infty)$ to be zero. Analysis shows that this is not a tenable hypothesis. We require ϕ and ϕ' to be always positive, otherwise reversed flow occurs. Likewise we expect ψ' to be positive everywhere, and so ψ must be negative since it is zero at infinity. Equation (3.3) now shows that ψ'' is negative, and hence from (3.2), ϕ''' is positive. Consequently ϕ'' increases steadily, and cannot be zero at infinity. Indeed this argument suggests that solutions satisfying (3.4) are possible only for sufficiently small values of $\beta\epsilon^{-\frac{1}{3}}$.

As $\beta\epsilon^{-\frac{1}{3}}$ rises to its limiting value (assuming that this exists), it is clear that $\phi''(0) \to 0$. We cannot take $\phi''(0) = 0$, as may be seen from the equation

$$\phi''' + \epsilon^{-1}\phi\phi'' - \psi^2\phi' + \psi\psi'\phi = 0, \tag{3.5}$$

obtained from (3.2) and (3.3). This equation and its derivatives show that if $\phi(0) = \phi'(0) = \phi''(0) = 0$, then $\phi \equiv 0$.

Suppose instead that $\phi''(0) = D$, a small constant, and that $\psi(0) = -C$. In a substantial region near the wall ϕ and ϕ' are then small, and from (3.3) so also is ψ''. Hence, approximately,

$$\psi = 2\theta - C, \quad \psi' = 2 \tag{3.6}$$

in this region and, with the term in ϵ^{-1} omitted, (3.5) becomes

$$\phi''' - (2\theta - C)^2\phi' + 2(2\theta - C)\phi = 0. \tag{3.7}$$

The required solution of (3.7) is found as

$$\phi'' = D\cosh(C\theta - \theta^2), \tag{3.8}$$

using the fact that $\phi = 2\theta - C$ is one solution of the equation, and hence, from (3.2), $\psi'' = -D\sinh(C\theta - \theta^2)$. Now ψ'' must be appreciable by the time ψ becomes small, near $\theta = \frac{1}{2}C$, or the boundary conditions $\psi(\infty) = 0$ cannot possibly be satisfied. Consequently if D is small, C must be large.

If we assume $C\theta - \theta^2$ to be large, (3.8) is approximately

$$\phi'' = \tfrac{1}{2}D\exp(C\theta - \theta^2) = E\exp(-t^2), \tag{3.9}$$

where

$$E = \tfrac{1}{2}D\exp(\tfrac{1}{4}C^2), \quad t = \theta - \tfrac{1}{2}C. \tag{3.10}$$

We take t as independent variable in place of θ. From (3.9) we then have

$$\phi' = E\operatorname{erfc}(-t), \quad \phi = E\{t\operatorname{erfc}(-t) + \tfrac{1}{2}\exp(-t^2)\}, \tag{3.11}$$

where

$$\operatorname{erfc} x = \int_x^\infty \exp(-u^2)\,du,$$

since the wall may be considered to be at $t = -\infty$. In terms of t, the expressions (3.6) are

$$\psi = 2t, \quad \psi' = 2. \tag{3.12}$$

The limiting form of our equations can now be seen. We have to integrate (3.2) (with the ϵ^{-1} term omitted) and (3.3), starting at large negative t with the values given by (3.9), (3.11) and (3.12). There is one disposable constant E, and this must be chosen to make $\psi(\infty) = 0$. The value of $\phi''(\infty)$ now determines the critical values of $\beta\epsilon^{-\frac{1}{3}}$.

Numerical integration showed clearly that only one value of E is acceptable. With E too small, ψ became positive, and with E too large, ψ' became negative while ψ was still negative. The values finally obtained were

$$E = 1{\cdot}091, \quad \phi''(\infty) = 1{\cdot}553, \tag{3.13}$$

which corresponds to

$$\beta = 0{\cdot}812\epsilon^{\frac{1}{3}}. \tag{3.14}$$

As β approaches this value there is an increasingly large region at the wall where there is effectively no flow and a constant magnetic field. For still larger β we may expect this to persist—a stagnant region of uniform field strength, separated from the main flow by a mixing region where the field falls to zero and the velocity rises to its free-stream value. In fact separation of the boundary layer has occurred.

For values of $\beta\epsilon^{-\frac{1}{3}}$ smaller than the separation value, but too great for the analysis of § 2 to be applicable, it would be necessary to solve (3.2) and (3.3) numerically, with the appropriate boundary conditions (3.4). There seems no reason to suppose that this family of solutions can be expressed in any simple manner.

4. Small conductivity, weak field

When ϵ is small, the viscous boundary layer is expected to be much thinner than the magnetic layer. In the viscous layer the appropriate equations are (2.1) and (2.2), the boundary conditions (2.3) having such modifications as prove to be necessary at $\eta = \infty$. In the magnetic layer the terms of (2.2) are comparable, and $f' = O(1)$ from the condition at $\eta = \infty$. For sufficiently weak fields the change in g' across the inner layer is small (as will be verified by the detailed solutions), and consequently we require $g' = O(1)$ in the outer layer. We therefore write, exactly as for the corresponding case in **G**,

$$\eta = \epsilon^{-\frac{1}{2}}\zeta, \quad f(\eta) = \epsilon^{-\frac{1}{2}}F(\zeta), \quad g(\eta) = \epsilon^{-\frac{1}{2}}G(\zeta), \tag{4.1}$$

and (2.1) and (2.2) become

$$\epsilon F''' + FF'' - \beta GG'' = 0, \tag{4.2}$$

$$G'' + FG' - F'G = 0. \tag{4.3}$$

Necessary boundary conditions are

$$F(0) = 0, \quad F'(\infty) = 2, \quad G(\infty) = 0. \tag{4.4}$$

(Without the first of these, the matching of the inner and outer solutions would demand an unacceptable value of $f(\infty)$.)

When the field is weak the first approximations $F = F_0$, etc., are found by putting $\epsilon = \beta = 0$. Equation (4.2) gives $F_0 = 2\zeta$, and (4.3) becomes

$$G_0'' + 2\zeta G_0' - 2G_0 = 0. \tag{4.5}$$

Equation (2.2) suggests that the change in g_0' across the inner layer is negligible; consequently $G_0'(0) = 2$ and the required solution of (4.5) is

$$G_0 = 2\pi^{-\frac{1}{2}}\{2\zeta\,\mathrm{erfc}\,\zeta - \exp(-\zeta^2)\}. \tag{4.6}$$

Thus, for ζ small,

$$G_0 \doteqdot -2\pi^{-\frac{1}{2}} + 2\zeta - 2\pi^{-\frac{1}{2}}\zeta^2 + \ldots, \tag{4.7}$$

which implies that, for η large,

$$g \sim -2\pi^{-\frac{1}{2}}\epsilon^{-\frac{1}{2}} + 2\eta - 2\pi^{-\frac{1}{2}}\epsilon^{\frac{1}{2}}\eta^2 + \ldots, \tag{4.8}$$

and so g must be $O(\epsilon^{-\frac{1}{2}})$ rather than $O(1)$ in the inner layer. The basic solution of (2.1) is $f_0 = B(\eta)$, the Blasius function as before. Equation (2.2) shows that $g'' = O(\epsilon)\,g$, and using (4.8) and the boundary condition (2.3) at $\eta = 0$ we obtain as our first approximation

$$g_0 = -2\pi^{-\frac{1}{2}}\epsilon^{-\frac{1}{2}} + (2\eta - c_0), \tag{4.9}$$

where c_0 is a constant to be determined later.

It might be thought more logical to replace $g(\eta)$ by a new function $\epsilon^{-\frac{1}{2}}h(\eta)$, say, so that h would be $O(1)$ in ϵ. Unfortunately this would create as many irregularities as it eliminated. The last term of (4.9) would then be relegated to the next approximation, yet it is needed to satisfy the boundary condition $g'(0) = 2$, and to fix $G_0(\zeta)$. In practice there is no special difficulty; there are as before just sufficient equations and boundary conditions to determine the next terms in the expansions at each stage.

We now consider improvements to these solutions. Contributions proportional to β are needed to satisfy terms in (2.1) and (4.2). If we write $F = F_0 + \beta F_1$, etc., we obtain from (4.2)

$$F_1'' = (4/\pi\zeta)\{\exp(-2\zeta^2) - 2\zeta\exp(-\zeta^2)\,\mathrm{erfc}\,\zeta\}, \tag{4.10}$$

and hence for ζ small

$$F_1'' \sim 4/\pi\zeta, \quad F_1' \sim (4/\pi)\log\zeta. \tag{4.11}$$

From equations (2.1) and (2.2),

$$f_1''' + Bf_1'' + B''f_1 = (4/\pi)B', \tag{4.12}$$

and hence for η large

$$f_1''' + (2\eta - c)f_1'' \sim 8/\pi,$$

since $B \sim 2\eta - c$, where $c = 1{\cdot}7208$, and so

$$f_1' \sim (4/\pi)\log\eta. \tag{4.13}$$

Now $\log\eta = \log\zeta + \frac{1}{2}\log(1/\epsilon)$, and we see (as in **G**) that the necessary matching of the inner and outer solutions can be achieved only if the expansions also contain terms in $\beta\log(1/\epsilon)$.

Let $F = F_0 + \beta F_1 + \beta \log(1/\epsilon) F_2$, etc. Then the matching requirement is that

$$F_2'(0) = f_2'(\infty) + 2/\pi. \tag{4.14}$$

The governing equations are

$$F_2''' = 0, \quad f_2'''' + Bf_2'' + B''f_2 = 0,$$

with boundary conditions

$$F_2(0) = 0, \quad F_2'(\infty) = 0, \quad f_2(0) = 0, \quad f_2'(0) = 0,$$

and the required solutions are

$$F_2 = 0, \quad f_2 = -(1/2\pi)(B + \eta B'). \tag{4.15}$$

No comparable logarithmic terms occur in g_1 or G_1, so $g_2 = G_2 = 0$.

We can now return to the equations for F_1, etc. Integration of (4.10) with the condition $F_1'(\infty) = 0$ gives

$$F_1' = -(2/\pi)\,\mathrm{ei}\,(2\zeta^2) + (4/\pi)\,\mathrm{erfc}^2\,\zeta, \tag{4.16}$$

where
$$\mathrm{ei}\,x = \int_x^\infty \frac{e^{-u}}{u}\,du.$$

For x small, $\mathrm{ei}\,x \doteq -\log x - \gamma$, where $\gamma = 0{\cdot}5772$ is Euler's constant, and hence for ζ small,

$$F_1' \doteq (4/\pi)\log\zeta + (2/\pi)\log 2 + (2\gamma/\pi) + 1 = (4/\pi)\log\zeta + 1{\cdot}809. \tag{4.17}$$

For matching we therefore require that

$$f_1' \sim (4/\pi)\log\eta + 1{\cdot}809 \tag{4.18}$$

for η large. Numerical integration of (4.12), satisfying (4.18) and the conditions $f_1(0) = f_1'(0) = 0$, gave the value

$$f_1''(0) = 0{\cdot}506. \tag{4.19}$$

Equation (2.2) and the boundary conditions show that $g_1 =$ constant, and so $G_1'(0) = 0$. The equation for G_1 is

$$G_1'' + 2\zeta G_1' - 2G_1 = F_1' G_0 - F_1 G_0', \tag{4.20}$$

and this may now be solved analytically, using previous results. It was found that $G_1(0) = -2{\cdot}209$ and hence

$$g_1 = -2{\cdot}209\epsilon^{-\frac{1}{2}} + c_1, \tag{4.21}$$

where c_1 is a constant of order unity.

The perturbations to the basic solution of order $\epsilon^{\frac{1}{2}}$, for which we write

$$F = F_0 + \epsilon^{\frac{1}{2}} F_3, \quad \text{etc.},$$

are found by a routine application of the methods described above and in **G**. The results obtained are

$$\left.\begin{aligned} F_3 = -c, \quad & G_3 = -2\pi^{-\frac{1}{2}} c\,\mathrm{erfc}\,\zeta, \\ f_3 = 0, \quad & g_3 = -2\pi^{-\frac{1}{2}}\eta^2 + 2\pi^{-\frac{1}{2}} c\eta + c_3, \end{aligned}\right\} \tag{4.22}$$

where c_3 is a constant which cannot be determined at this stage and (as before) $c = 1{\cdot}7208$. During the calculation it is proved that, in equation (4.9), $c_0 = c$.

As in § 2, there is no sign of any bar to continuing with the calculation of further terms in the expansions, if desired.

To summarize, the chief results obtained in this section, corresponding to (2.28), (2.29) and (2.30) are

$$\begin{aligned}\tau_w &= \tfrac{1}{4}\rho(U_0^3\nu/x)^{\frac{1}{2}}f''(0)\\ &= 0{\cdot}33206\rho(U_0^3\nu/x)^{\frac{1}{2}}\{1-0{\cdot}477\beta\log(1/\epsilon)+0{\cdot}381\beta+\ldots\}, \quad (4.23)\\ H_n &= -\tfrac{1}{2}H_0(\nu/U_0x)^{\frac{1}{2}}g(0)\\ &= 0{\cdot}5642H_0(\sigma\mu U_0x)^{-\frac{1}{2}}\{1+1{\cdot}958\beta+1{\cdot}525\epsilon^{\frac{1}{2}}+\ldots\}, \quad (4.24)\\ T &= 0{\cdot}33206\rho(U_0^3\nu/x)^{\frac{1}{2}}\{1-0{\cdot}477\beta\log(1/\epsilon)+2{\cdot}080\beta+\ldots\}. \quad (4.25)\end{aligned}$$

For these results to be of value it is necessary that β shall be small compared with $\{\log(1/\epsilon)\}^{-1}$, which is itself small. In most physical situations so far examined ϵ is in fact very small, peraphs roughly of the order of 10^{-5}. In this case β would have to be less than 0·2, if the second term in the braces in (4.23) is not to exceed the first. As was the case for large ϵ, a separate investigation is needed to determine what happens if $\beta\log(1/\epsilon)$ is not small.

5. Small conductivity, strong field

In this section we shall show that when ϵ is small, separation occurs when β reaches a certain multiple of $\{\log(1/\epsilon)\}^{-1}$. This is as might have been forecast from (4.23). The analysis is more complicated than that in § 3 for the case of ϵ large, due chiefly to the presence of the logarithmic terms. In presenting the results we shall set out the forms of transformation required in the inner and outer layers, and proceed to derive the relevant solutions. We shall then look back and, in the light of the results obtained, shall be able to perceive the necessity for the assumptions made previously.

In the inner layer we write

$$\eta = \beta_0^{-\frac{1}{2}}\theta, \quad f(\eta) = \beta_0^{\frac{1}{2}}\phi(\theta), \quad g(\eta) = \epsilon^{-\frac{1}{2}}\{\log(1/\epsilon)\}^{-\frac{1}{2}}\beta^{-\frac{1}{2}}\beta_0^{\frac{1}{2}}\psi(\theta), \qquad (5.1)$$

and obtain from (2.1) and (2.2)

$$\phi'''+\phi\phi''-\{\log(1/\epsilon)\}^{-1}(\phi'\psi^2-\phi\psi\psi') = 0, \qquad (5.2)$$

$$\psi''+\epsilon(\phi\psi'-\phi'\psi) = 0, \qquad (5.3)$$

with boundary conditions

$$\phi(0) = \phi'(0) = 0, \quad \psi(0) = -1. \qquad (5.4)$$

The boundary condition on $g(\infty)$ from (2.3) is ignored as before, and the inner boundary condition is applied to ψ rather than ψ'. The new constant β_0 (assumed to be $O(1)$, as will be justified *a posteriori*) is chosen so that $\psi(0) = -1$. The choice of symbol is influenced by the fact that β_0 is of the same general order of magnitude as β, as will appear below. In the outer layer we write

$$\eta = \epsilon^{-\frac{1}{2}}\zeta, \quad f(\eta) = \epsilon^{-\frac{1}{2}}F(\zeta), \quad g(\eta) = \epsilon^{-\frac{1}{2}}\{\log(1/\epsilon)\}^{-\frac{1}{2}}\beta^{-\frac{1}{2}}G(\zeta), \qquad (5.5)$$

and obtain

$$\epsilon F''''+FF''-\{\log(1/\epsilon)\}^{-1}GG'' = 0, \qquad (5.6)$$

$$G''+FG'-F'G = 0, \qquad (5.7)$$

with boundary conditions

$$F(0) = 0, \quad F'(\infty) = 2, \quad G(\infty) = 0. \tag{5.8}$$

We note, for purposes of matching, that

$$F' = \beta_0 \phi', \quad G = \beta_0^{\frac{1}{2}} \psi, \tag{5.9}$$

in their common region of validity.

We now look for solutions of (5.2), (5.3), (5.6) and (5.7) in the form

$$\phi = \phi_0 + \{\log(1/\epsilon)\}^{-1} \phi_1 + \ldots, \tag{5.10}$$

etc., where the functions ϕ_0, etc., may depend upon β_0, but not on β or ϵ explicitly. From (5.3) and the boundary conditions,

$$\psi_0 = -1, \quad \psi_1 = 0. \tag{5.11}$$

Terms linear in θ are forbidden by the matching requirement. Equation (5.2) gives $\phi_0''' + \phi_0 \phi_0'' = 0$, and hence

$$\phi_0 = \alpha B(\alpha\theta), \tag{5.12}$$

where α is some constant, and so

$$\phi_0'(\infty) = 2\alpha^2. \tag{5.13}$$

The next most important terms in (5.2) require

$$\phi_1''' + \phi_0 \phi_1'' + \phi_0'' \phi_1 = \phi_0', \tag{5.14}$$

which implies (as in equation (4.12)) that, for θ large,

$$\phi_1' \sim \log\theta = \log\zeta + \tfrac{1}{2}\log(1/\epsilon) + \tfrac{1}{2}\log\beta_0. \tag{5.15}$$

From (5.13) and (5.15), for θ large,

$$\phi' \sim 2\alpha^2 + \tfrac{1}{2} + \{\log(1/\epsilon)\}^{-1}\log\zeta + \ldots. \tag{5.16}$$

From (5.6) and (5.8) we obtain

$$F_0 = 2\zeta, \quad G_0 = \{2\zeta \operatorname{erfc}\zeta - \exp(-\zeta^2)\}\beta_0^{\frac{1}{2}}, \tag{5.17}$$

as in (4.6), since from (5.9) and (5.11), $G_0(0) = -\beta_0^{\frac{1}{2}}$. The next terms in (5.6) require

$$2\zeta F_1'' = G_0 G_0''. \tag{5.18}$$

As $\zeta \to 0$, $G_0 G_0'' \to 2\beta_0$, which implies that

$$F_1' \sim \beta_0 \log\zeta. \tag{5.19}$$

Thus, for ζ small,

$$F' \sim 2 + \{\log(1/\epsilon)\}^{-1}\beta_0 \log\zeta + \ldots. \tag{5.20}$$

Comparing (5.16) and (5.20), we see from (5.9) that, for matching of the inner and outer solutions,

$$(2\alpha^2 + \tfrac{1}{2})\beta_0 = 2. \tag{5.21}$$

Since $\alpha^2 > 0$, this can be satisfied only if $\beta_0 < 4$. As $\beta_0 \to 4$, $\alpha \to 0$, and hence $\phi_0 \to 0$. It is clear that separation occurs, of the same type as in § 3. The viscous layer has been brought to rest, and the whole velocity change takes place within the magnetic layer.

It remains to relate β_0 to β. From (5.3), g' does not change significantly across the inner layer and hence from (5.5) and (5.17), to a first approximation,

$$g'(0) = \pi^{\frac{1}{2}}\beta_0^{\frac{1}{2}}\{\log(1/\epsilon)\}^{-\frac{1}{2}}\beta^{-\frac{1}{2}}. \tag{5.22}$$

The original boundary condition (2.3) was $g'(0) = 2$; this shows that we require

$$\beta_0 = (4/\pi)\,\beta\log(1/\epsilon). \tag{5.23}$$

Thus the separation value $\beta_0 = 4$ corresponds to

$$\beta = \pi\{\log(1/\epsilon)\}^{-1}. \tag{5.24}$$

Use of (5.1), (5.12), (5.21) and (5.23) shows that the first approximation to the skin-friction is given by

$$\frac{\tau_w}{\rho}\left(\frac{x}{U_0^3\nu}\right)^{\frac{1}{2}} = \tfrac{1}{4}\beta_0^{\frac{3}{2}}\phi_0''(0) = 0{\cdot}33206\left(1-\frac{\beta}{\pi}\log\frac{1}{\epsilon}\right)^{\frac{3}{2}}. \tag{5.25}$$

The first two terms in the binomial expansion of (5.25) are in agreement with (4.23). The corresponding value for H_n, obtained from (5.11), is identical with the first term of the series (4.24).

Let us now review the analysis of this section. Following the ideas of § 3, it at first seems natural to choose a transformation of the variables which omits the factors $\{\log(1/\epsilon)\}^{-\frac{1}{2}}$ in the definitions (5.1) and (5.5), so that the factors $\{\log(1/\epsilon)\}^{-1}$ do not appear in (5.2) and (5.6). It may be verified that in this case the matching condition (5.21) does not contain the term $2\alpha^2$ and has zero on the right-hand side, so it cannot be satisfied with $\beta_0 \neq 0$. The next attempt might be to introduce the factor $\{\log(1/\epsilon)\}^{-1}$ in one only of the equations (5.2) and (5.6). If it is present in (5.2) only, the right-hand side of (5.21) is still zero and the equation remains insoluble. If it is present in (5.6) only, (5.21) becomes $\beta_0 = 4$. This form of the equations could, therefore, apply only in this limiting case, and detailed inspection shows that even when there remain unsatisfactory features. The transformation as originally described is the remaining hope, and the fact that the solution proceeds smoothly gives confidence in its correctness.

To sum up the results of this paper, in § 2 and § 4 solutions in series are developed for the velocity and magnetic fields, valid when β is sufficiently small, and when ϵ is either large or small. In § 3 and § 5 it is shown that for both large and small ϵ, separation occurs when β reaches a critical value. In each case the breakdown of the flow takes the form of the boundary layer moving away from the surface, leaving a dead-air region of uniform magnetic field strength. The magnetic drag remains non-zero, even at separation. It is natural to conclude that separation of this type occurs for sufficiently strong applied fields for any value of the conductivity.

Physically, the separation may be attributed to the field lines which emerge from the plate to supply the magnetic flux in the boundary layer, for fluid near the plate experiences a resistance in forcing its way past these field lines.

The situation has some resemblance to the non-conducting boundary-layer flow past a plate with emission of fluid at the surface (with velocity proportional to $x^{-\frac{1}{2}}$ so as to give a similarity solution). If the emission velocity exceeds a certain critical value, the whole boundary layer leaves the surface and is separated from it by a region of stagnant fluid, as in the present case.

Appendix. Errata in Glauert (1961)

Page 277, equation (1.5): for $(f-\eta f')$ read $(\eta f'-f)$.
Page 277, equation (1.6): for $(g-\eta g')$ read $(\eta g'-g)$.
Page 281, boundary conditions between (3.30) and (3.31):
for $q_2(0)=0$ read $q_2-\frac{2}{3}\theta^{-1}\to 0$ as $\theta\to 0$;
for $P_2''(\infty)=c_2$ read $P_2''+\frac{1}{6}A^2\xi^3-\frac{4}{3}A\,\dfrac{\beta}{1-\beta}\log\xi\to c_2$ as $\xi\to\infty$;
for $Q_2''(\infty)=d_2$ read $Q_2''+\frac{1}{6}A^2\xi^3-\left(\dfrac{4}{3}\dfrac{\beta}{1-\beta}+\dfrac{8}{15}\right)A\log\xi\to d_2$ as $\xi\to\infty$.
Page 286, line after (6.2): for $\beta>1$ read $\beta<1$.
Page 287, equation (A 4): delete $\operatorname{div}\mathbf{E}=0$.

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